

Compactly generated rectifiable spaces or paratopological groups*

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Abstract. A rectifiable space (or a paratopological group) G is compactly generated if $G = \langle K \rangle$ for some compact subset K of G . In this paper, we mainly discuss compactly generated rectifiable spaces or paratopological groups. The main results are that: (1) each σ -compact metrizable rectifiable space containing a dense compactly generated rectifiable subspace is compactly generated; (2) a metrisable rectifiable space is compactly generated if and only if it is σ -compact and finitely generated modulo open sets; (3) any σ -compact paratopological group can be embedded as a closed paratopological subgroup in some compactly generated paratopological group. Finally, we consider generalized metric properties of compactly generated rectifiable spaces.

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1. Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product map from $G \times G$ onto G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous. A topological space G is said to be a *rectifiable space* [4] provided that there are a surjective homeomorphism $\varphi : G \times G \rightarrow G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \rightarrow G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G . It is well known that rectifiable spaces and paratopological groups are all good generalizations of topological groups. It is easy to see that a topological group G with the neutral element e has a rectification $\varphi(x, y) = (x, x^{-1}y)$. However, there exists a paratopological group which is not a rectifiable space; Sorgenfrey line ([8, Example 1.2.2]) is such an example. Also, the 7-dimensional sphere S_7 is rectifiable but not a topological group [21, § 3]. In fact, it

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is even not a semitopological group, because each (locally) compact semitopological group is a topological group [7]. Further, it is easy to see that both paratopological groups and rectifiable spaces are homogeneous.

Recently, the study of rectifiable spaces has become an interesting topic in topological algebra, see [1, 11, 13, 14, 15, 16, 20, 21].

2. Preliminaries

The following theorem was announced for the first time in [4], and the readers can see the proof in [5, 11, 20].

Theorem 1 (see [4]). *A topological space G is rectifiable if and only if there exist an element $e \in G$ and two continuous maps $p : G^2 \rightarrow G$, $q : G^2 \rightarrow G$ such that for any $x \in G, y \in G$ the next identities hold:*

$$p(x, q(x, y)) = q(x, p(x, y)) = y, q(x, x) = e.$$

In fact, we can assume that $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$ in Theorem 1. If we fix a point $x \in G$, then $f_x, g_x : G \rightarrow G$ defined with $f_x(y) = p(x, y)$ and $g_x(y) = q(x, y)$, for each $y \in G$, are homeomorphisms. We denote f_x, g_x by $p(x, G), q(x, G)$, respectively.

If G is a rectifiable space, then we shall call the map p the multiplication on G . Moreover, sometimes we shall write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$; since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x . Hence a rectifiable space G is a topological algebraic system with binary operations p, q , 0-ary operation e and identities as above. It is easy to see that this algebraic system need not satisfy the associative law about the multiplication operation p . Clearly, every topological loop is rectifiable.

If G is a rectifiable space (or a paratopological group) and $X \subset G$, then we use $\langle X \rangle$ to denote the smallest rectifiable subspace of G which contains X . A set X algebraically generates G if $G = \langle X \rangle$.

Recall that a rectifiable space G (a paratopological group) is:

- (1) σ -compact if $G = \bigcup \{K_n : n \in \mathbb{N}\}$, where each K_n is compact, and
- (2) compactly generated if $G = \langle K \rangle$ for some compact subset K of G .

Note 1. (a): Obviously, each compactly generated rectifiable space is σ -compact. However, there exists a compactly generated paratopological group which is not σ -compact. Indeed, let X be an uncountable compact space, and let $AP(X)$ be a free Abelian paratopological group. Then $-X$ is closed discrete in $AP(X)$ [17], which implies that $AP(X)$ is not σ -compact. Moreover, $AP(X)$ is not a topological group.

(b): There exists a countable, metrizable, and compactly generated paratopological group which is not a topological group. Indeed, let the rational number \mathbb{Q} with the subspace topology of Sorgenfrey line. Then \mathbb{Q} is a countable, metrizable paratopological group which is not a topological group. Put $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$; then $Q = \langle S \rangle$. Therefore, \mathbb{Q} is compactly generated.

(c): Sorgenfrey line is not a compactly generated paratopological group since each compact subset of Sorgenfrey line is countable [2, 3.3.b].

All spaces considered in this paper are supposed to be T_1 and regular unless stated otherwise. The notation \mathbb{N} denotes the set of all positive integer numbers. The letter e denotes the neutral element of a group or the right neutral element of a rectifiable space. Readers may refer to [2, 8, 10] for notations and terminology not explicitly given here.

3. Compactly generated rectifiable spaces

In this section, we mainly discuss compactly generated rectifiable spaces. Firstly, we give some technical lemmas.

Lemma 1 (see [9]). *Let $\{U_n : n \in \mathbb{N}\}$ be a local base at point e of a topological space G such that $\overline{U_{n+1}} \subset U_n$ for all $n \in \mathbb{N}$. Assume that $\{F_n : n \in \mathbb{N}\}$ is a sequence of subsets of G such that*

1. *each F_n is compact, and*
2. *$F_n \subset \overline{U_n}$.*

Then $K = \bigcup\{F_n : n \in \mathbb{N}\} \cup \{e\}$ is compact. Moreover, if each F_n is finite, then for each enumeration $i : \mathbb{N} \rightarrow K$ a sequence $\{i(n) : n \in \mathbb{N}\}$ converges to e .

Let A be a subspace of a rectifiable space G . Then A is called a *rectifiable subspace* [14] of G if we have $p(A, A) \subset A$ and $q(A, A) \subset A$.

Lemma 2 (see [14]). *Let G be a rectifiable space. If V is an open rectifiable subspace of G , then V is closed in G .*

Lemma 3. *Let H be a dense rectifiable subspace of a rectifiable space G . Then for each open rectifiable subspace E of H there exists an open rectifiable subspace E' of G such that $E' \cap H = E$.*

Proof. Let

$$E' = \bigcup\{V : V \text{ is open in } G \text{ and } \text{cl}_G(V) \cap H \subset E\}.$$

Obviously, E' is an open subset of G and $E' \cap H = E$. Now, we shall prove that E' is a rectifiable subspace of G .

Indeed, suppose that $a, b \in E'$. It follows from the definition of E' that there exist open sets U and V in G such that $a \in U, b \in V, \text{cl}_G(U) \cap H \subset E$ and $\text{cl}_G(V) \cap H \subset E$. By the density of H in G , we have $\text{cl}_G(U \cap H) = \text{cl}_G(U)$ and $\text{cl}_G(V \cap H) = \text{cl}_G(V)$. Therefore, it follows from the continuity of p in G that

$$p(U, V) \subset p(\text{cl}_G(U), \text{cl}_G(V)) = p(\text{cl}_G(U \cap H), \text{cl}_G(V \cap H)) \subset \text{cl}_G(p(U \cap H, V \cap H)).$$

Then we have $\text{cl}_G(p(U, V)) = \text{cl}_G(p(U \cap H, V \cap H))$, and

$$\text{cl}_G(p(U, V)) \cap H = \text{cl}_G(p(U \cap H, V \cap H)) \cap H \subset \text{cl}_G(p(E, E)) \cap H = \text{cl}_G(E) \cap H = E$$

since E is closed in H by Lemma 2. Therefore, $p(a, b) \in p(U, V) \subset E'$.

Suppose that $c, d \in E'$. Then there exist open sets O, W in G such that $c \in O, d \in W, \text{cl}_G(O) \cap H \subset E$ and $\text{cl}_G(W) \cap H \subset E$. Obviously, $q(O, W)$ is open in G . Moreover, it is also easy to see that $\text{cl}_G(q(O, W)) = \text{cl}_G(q(O \cap H, W \cap H))$. Since

$$\text{cl}_G(q(O, W)) \cap H = \text{cl}_G(q(O \cap H, W \cap H)) \cap H \subset \text{cl}_G(q(E, E)) \cap H \subset \text{cl}_G(E) \cap H = E,$$

it follows that $q(c, d) \in q(O, W) \subset E'$. \square

Corollary 1. *A dense rectifiable subspace of a connected rectifiable space has no proper open rectifiable subspaces.*

Proof. By Lemma 2, each open rectifiable subspace of a rectifiable space is closed, and therefore, a connected rectifiable space cannot have proper open rectifiable subspaces. Now the result follows from Lemma 3. \square

Lemma 4 (see [14]). *Let G be a rectifiable space. If Y is a dense subset of G and U is an open neighborhood of the right neutral element e of G , then $G = Y \cdot U$.*

Theorem 2. *If a σ -compact metrizable rectifiable space G contains a dense compactly generated rectifiable subspace, then G is also compactly generated.*

Proof. Let H be a dense rectifiable subspace of G such that H is generated by some compact set E , and let $G = \bigcup \{K_n : n \in \mathbb{N}\}$, where each K_n is compact. Since G is metrizable, the point e has a countable local base $\{U_n : n \in \mathbb{N}\}$, where $\overline{U_{n+1}} \subset U_n$ for each $n \in \mathbb{N}$. By the density of H in G , it follows from Lemma 4 that $p(H, U_n) = G$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a finite subset F_n of H such that $K_n \subset p(F_n, U_n)$, and put $L_n = \overline{U_n} \cap q(F_n, K_n)$, then each $K_n \subset p(F_n, L_n)$ since $K_n \subset p(F_n, q(F_n, K_n))$. Obviously, each L_n is compact and, by Lemma 1, $L = \bigcup \{L_n : n \in \mathbb{N}\}$ is also compact. Therefore,

$$G = \bigcup \{K_n : n \in \mathbb{N}\} \subset \bigcup \{p(F_n, L_n) : n \in \mathbb{N}\} \subset \bigcup \{p(H, L_n) : n \in \mathbb{N}\} \subset p(H, L).$$

Since H is generated by E , G is generated by the compact set $E \cup L$. Therefore, G is compactly generated. \square

Corollary 2. *If a σ -compact metrizable rectifiable space G contains a dense finitely generated rectifiable subspace, then G is also compactly generated.*

Next, we define the notion of finitely generated modulo open sets in rectifiable spaces which contains all compactly generated rectifiable spaces.

Definition 1. *We will say that a rectifiable space (or a paratopological group) G is finitely generated modulo open sets if for each non-empty open rectifiable subspace H of G there exists a finite subset F of G such that $G = \langle F \cup H \rangle$.*

Proposition 1. *Let G be a rectifiable space. Then the following conditions are equivalent:*

1. G is finitely generated modulo open sets;

2. for each non-empty open subset V of G there exists a finite subset F of G such that $G = \langle F \cup V \rangle$.

Proof. Obviously, $(2) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. Let V be a non-empty open subset V of G . Let H be the rectifiable subspace generated by V , that is, $H = \langle V \rangle$. Obviously, H is open in G , and so by (2) there is a finite set $F \subset G$ such that

$$G = \langle F \cup H \rangle = \langle F \cup \langle V \rangle \rangle = \langle F \cup V \rangle.$$

□

Theorem 3. *If a rectifiable space G is compactly generated, then it is finitely generated modulo open sets.*

Proof. Assume that $G = \langle K \rangle$, where K is a compact set. Let H be an open rectifiable subspace of G . Then $\mathcal{H} = \{g \cdot H : g \in G\}$ is an open covering of G . Since K is compact, there exist finitely many elements of \mathcal{H} , say $g_1 \cdot H, \dots, g_n \cdot H$, which cover K . Put $F = \{g_1, \dots, g_n\}$. Then $G = \langle F \cup H \rangle$. □

Theorem 4. *Let G be a metrizable rectifiable space G and A a countable subset of G . Suppose that G is finitely generated modulo open sets. Then G contains a sequence \mathcal{S} converging to e of G such that $A \subset \langle \mathcal{S} \rangle$.*

Proof. Let $A = \{a_n : n \in \omega\}$. Since G is metrizable, let $\{U_n : n \in \omega\}$ be a local base at e such that

$$G = U_0 \supseteq U_1 \supseteq \dots \supseteq U_n \supseteq \dots$$

Since G is finitely generated modulo open sets, for each $n \in \omega$ we can fix a finite set F_n such that $G = \langle F_n \cup U_{n+1} \rangle$.

By induction on n , we will define a sequence $\{B_n : n \in \omega\}$ of finite subsets of G with the following properties:

- (a) $B_n \subset U_n$;
- (b) $G = \langle B_0 \cup B_1 \cup \dots \cup B_n \cup U_{n+1} \rangle$, and
- (c) $a_n \in \langle B_0 \cup B_1 \cup \dots \cup B_n \rangle$.

To begin with, let $B_0 = F_0 \cup \{a_0\}$; then B_0 satisfies all three conditions (a)-(c). Assume that we have already defined finite sets B_0, B_1, \dots, B_{n-1} satisfying all three conditions (a)-(c). By (b), $F_n \cup \{a_n\} \subset \langle B_0 \cup B_1 \cup \dots \cup B_{n-1} \cup U_n \rangle$. Since F_n is finite, we can find a finite set $B_n \subset U_n$ such that

$$F_n \cup \{a_n\} \subset \langle B_0 \cup B_1 \cup \dots \cup B_{n-1} \cup B_n \rangle.$$

Clearly, (a)-(c) are satisfied.

Put $\mathcal{S} = \cup\{B_n : n \in \omega\}$. By (c), $A \subset \langle \mathcal{S} \rangle$. By Lemma 1 and (a), \mathcal{S} can be enumerated as a sequence converging to e . □

Theorem 5. *Let G be a σ -compact metrizable rectifiable space G . Then G is compactly generated if and only if G is finitely generated modulo open sets.*

Proof. By Theorem 3, we only need to prove the sufficiency. Suppose that for each open rectifiable subspace H of G there exists a finite set F such that $G = \langle F \cup H \rangle$. Obviously, G is separable, and let D be a countable dense subset of G . By Theorem 4, G has a dense compactly generated rectifiable subspace, and by Theorem 2, G is compactly generated. \square

Corollary 3. *A metrizable rectifiable space G is compactly generated if and only if G is σ -compact and finitely generated modulo open sets.*

A rectifiable space without proper open rectifiable subspaces trivially satisfies condition (2) of Proposition 1. Therefore, we have the following corollary.

Corollary 4. *A σ -compact metrizable rectifiable space G without proper open rectifiable subspaces is compactly generated.*

By Corollaries 1 and 4, we also have the following corollary.

Corollary 5. *A σ -compact dense rectifiable subspace of a connected metrizable rectifiable space G is compactly generated.*

Corollary 6. *A σ -compact connected metrizable rectifiable space G is compactly generated.*

By Theorems 3 and 4, it is easy to prove the following theorem.

Theorem 6. *A countable metrizable rectifiable space is compactly generated if and only if it is compactly generated by a sequence converging to the right neutral element e .*

4. Compactly generated paratopological groups

In this section, we mainly discuss compactly generated paratopological groups.

Lemma 5. *Let G be a paratopological group. If Y is a dense subset of G and U is an open neighborhood of the neutral element e of G , then $G = Y^{-1} \cdot U$.*

Proof. For arbitrary $g \in G$, since Y is a dense subset of G , we have $Ug^{-1} \cap Y \neq \emptyset$. Take $x \in Ug^{-1} \cap Y$. Then $g \in x^{-1}U \subset Y^{-1} \cdot U$. \square

The proof of the following theorem is similar to that of Theorem 2.

Theorem 7. *If a σ -compact first-countable paratopological group G contains a dense compactly generated subgroup, then G is also compactly generated.*

Proof. Let H be a dense subgroup of G such that H is generated by some compact set E , and let $G = \bigcup \{K_n : n \in \mathbb{N}\}$, where each K_n is compact. Since G is first-countable, the point e has a countable local base $\{U_n : n \in \mathbb{N}\}$, where $\overline{U_{n+1}} \subset U_n$ for each $n \in \mathbb{N}$. By the density of H in G , it follows from Lemma 5 that $HU_n = G$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a finite subset F_n of H such that $K_n \subset F_n U_n$, and put $L_n = \overline{U_n} \cap (F_n)^{-1} K_n$, then each $K_n \subset F_n L_n$ since $K_n \subset$

$F_n(F_n)^{-1}K_n$. Obviously, each L_n is compact and, by Lemma 1, $L = \bigcup\{L_n : n \in \mathbb{N}\}$ is also compact. Therefore,

$$G = \bigcup\{K_n : n \in \mathbb{N}\} \subset \bigcup\{F_n L_n : n \in \mathbb{N}\} \subset \bigcup\{H L_n : n \in \mathbb{N}\} \subset H L.$$

Since H is generated by E , G is generated by the compact set $E \cup L$. Therefore, G is compactly generated. \square

Note 2. Under the class of paratopological groups, we can obtain all results from Proposition 1 to Theorem 5 and Corollary 4 to Theorem 6 in Section 3 by similar proofs. In fact, the respective counterparts also hold for first-countable paratopological groups and this condition is weaker than the metrizability.

Since a compactly generated rectifiable space G is σ -compact, G has Souslin property, see [18] or [19]. Moreover, E.A. Reznichenko showed that every σ -compact Hausdorff paratopological group has Souslin property, see [2, Theorem 5.7.12]. However, the following question is still open.

Question 1. Let G be a compactly generated paratopological group. Does G have Souslin property?

Theorem 8. Any σ -compact paratopological group G can be embedded as a closed paratopological subgroup in some compactly generated paratopological group.

Proof. Let $\sigma\Pi = \sigma\Pi\{G_n : n \in \mathbb{Z}\}$ be the σ -product of copies of G with the topology induced from Tikhonov power $G^{\mathbb{Z}}$, where $\sigma\Pi$ is a σ -product with the neutral element e as a distinguished point. Then $\sigma\Pi$ is also a paratopological group. For each $n \in \mathbb{Z}$, let $i_n : G \rightarrow G_n$ be a topological isomorphism, and we can identify G_n with its image in $\sigma\Pi$ under the natural embedding. Suppose that $G = \bigcup\{K_n : n \in \mathbb{Z}\}$, where each K_n is compact. Let K denote the subspace $\bigcup_{n \in \mathbb{Z}} i_n(K_n)$ of the paratopological group $\sigma\Pi$. Since K is closed in the compact subspace $\Pi\{K_n : n \in \mathbb{Z}\}$ of the paratopological group $G^{\mathbb{Z}}$ under the natural embedding $\sigma\Pi \rightarrow G^{\mathbb{Z}}$, K is compact in $\sigma\Pi$.

The group \mathbb{Z} of integers with the discrete topology acts on the paratopological group $\sigma\Pi$ by shifting coordinates: for $x = (x_n)_{n \in \mathbb{Z}} \in \sigma\Pi$ and $k \in \mathbb{Z}$, $k \cdot x$ is the element of $\sigma\Pi$ whose n -th coordinate is x_{n+k} . Let G' denote the semidirect product $\sigma\Pi \rtimes \mathbb{Z}$. Assume 1_z is the smallest positive element of \mathbb{Z} . Then the space $K \cup \{1_z\}$ is a compact subspace of G' and $G' = \langle K \cup \{1_z\} \rangle$. Indeed, let $H = \langle K \cup \{1_z\} \rangle$ in G' . Clearly, $\mathbb{Z} \subset H$. Next, we shall prove that, for each $m \in \mathbb{Z}$, $G_m \subset H$. Take arbitrary $x \in G_m$. Then $i_m^{-1}(x) \in K_n$ for some $n \in \mathbb{Z}$. Let a be the element $(i_n i_m^{-1}(x), 0)$ and b the element $(e, m - n)$ of the semidirect product G' . Clearly, $a \in K \subset H$ and $b \in H$, and hence ba belongs to H . However, it is easy to see that $ba = x$. \square

If G be countable, then each of the sets K_n can be assumed finite. A simple analysis of the topological structure of the space $K \cup \{1_z\}$ enables us to obtain

Theorem 9. Any countable paratopological group G can be embedded as a closed paratopological subgroup in some paratopological group algebraically generated by a subspace homeomorphic to the one-point compactification $\partial\mathbb{N}$ of a countable discrete space.

Question 2. Can any σ -compact rectifiable space G be embedded as a closed rectifiable subspace in some compactly generated rectifiable space?

5. Generalized metrizability properties of compactly generated rectifiable spaces

A closed mapping f is called *perfect* if each fiber is compact.

Proposition 2. *Suppose that F is a compact subspace of a rectifiable space G . Then the restriction p and q to the subspace $F \times G$ is a perfect and open mapping of $F \times G$ onto G .*

Proof. We firstly prove that the restriction p to the subspace $F \times G$ is a perfect and open mapping of $F \times G$ onto G .

Let $f : F \times G \rightarrow F \times G$ be defined by $f(x, y) = (x, p(x, y))$ for each $(x, y) \in F \times G$. Obviously, f is continuous, one-to-one, and $f(F \times G) = F \times G$. Moreover, $f^{-1}(x, y) = (x, q(x, y))$. Therefore, f^{-1} is also continuous. Thus f is a homeomorphism. For $i = 1, 2$, denote by π_i the projection of $F \times G$ onto the i -th factor. Since $p(x, y) = \pi_2(x, p(x, y)) = \pi_2 f(x, y)$ for all $x \in F$ and $y \in G$, p is the composition of f and π_2 , that is, $p = \pi_2 \circ f$. Since F is compact, it follows from [8, Theorem 3.1.16] that π_2 is closed. Then p is closed since f is a homeomorphism and π_2 is closed. For each $y \in G$, $p^{-1}(y) = f^{-1}(F \times \{y\}) = \bigcup \{(x, q(x, y)) : x \in F\}$ is closed in the compact subspace $F \times q(F, y)$. Indeed, let $(x, q(z, y)) \in (F \times q(F, y)) \setminus p^{-1}(y)$, where $x, z \in F$. Then $q(x, y) \neq q(z, y)$, and thus there exist two open sets U and V in G such that $q(x, y) \in U$, $q(z, y) \in V$ and $U \cap V = \emptyset$. Since q is continuous, there exists an open neighborhood W of e such that $q(x \cdot W, y \cdot W) \subset U$ and $q(z \cdot W, y \cdot W) \subset V$. Then $(x \cdot W, V)$ is an open neighborhood of $(x, q(z, y))$. However, since $q(x \cdot w, y) \subset U$ for each $w \in W$, it follows that $(x \cdot W, V) \cap p^{-1}(y) = \emptyset$. Therefore, $p^{-1}(y)$ is closed in $F \times q(F, y)$, and thus it is compact. Then p is perfect.

Let O be an open subset of $F \times G$. Put $O' = \pi_1(O)$. For each $x \in O'$, let $U_x = \{y \in G : (x, y) \in O\}$; then O_x is open in G as the projection of the open subset $O \cap \pi_1^{-1}(x)$ of $\{x\} \times G$ onto the second factor. Therefore, $p(O) = \bigcup_{x \in O'} p(x, O_x)$ is open in G , which implies that p is an open mapping.

As for the mapping q , we only redefine the mapping f by $(x, y) = (x, q(x, y))$ for each $(x, y) \in F \times G$, and the rest of the proof is immediate. \square

Corollary 7. *Suppose that F is a compact subspace of a rectifiable space G , and that M is a closed subspace of G . Then $p(F, M)$ and $q(F, M)$ are all closed in G .*

Note 3. *Corollary 7 gives an affirmative answer to the following question. Recently, L.X. Peng and S.J. Guo [16] have also obtained Corollary 7. However, we prove Corollary 7 by a different method.*

Question 3 (see [15]). *Let G be a rectifiable. If F, P are compact and closed subsets of G , respectively, is $P \cdot F$ or $F \cdot P$ closed in G ?*

Since the restriction of a perfect mapping to a closed subspace is again a perfect mapping, it follows from Corollary 7 and Proposition 2 that we have the following corollary.

Corollary 8. *Suppose that F is a compact subspace of a rectifiable space G , and that M is a closed subspace of G . Then the restriction p and q to the subspace $F \times M$ is a perfect mapping of $F \times M$ onto a closed subspace of G .*

A space G is of *countable tightness* if for each subset A of G and each point $x \in \text{cl}(A)$ there exists a countable subset D of A such that $x \in \text{cl}(D)$.

Theorem 10. *Suppose that F is a compact subspace of a rectifiable space G and that M is a closed subspace of G . Suppose also that both F and M have countable tightness. Then both spaces $p(F, M)$ and $q(F, M)$ have countable tightness, too.*

Proof. Since perfect mappings do not increase the tightness and the tightness of the product $F \times M$ is countable by [8, 3.12.8(a)], it follows from Corollary 8 that both spaces $p(F, M)$ and $q(F, M)$ have countable tightness, too. \square

Theorem 11. *Suppose that F is a compact metrizable subspace of a rectifiable space G , and that M is a closed metrizable subspace of G . Then both spaces $p(F, M)$ and $q(F, M)$ are metrizable, too.*

Proof. By Corollary 7, $p(F, M)$ and $q(F, M)$ are closed in G . Since perfect mappings preserve the metrizability [8, Theorem 4.4.15], it follows from Corollary 8 that $p(F, M)$ and $q(F, M)$ are metrizable. \square

A *network* for a space X is a collection \mathcal{F} of subsets of X such that whenever $x \in U$ with U open, there exists $F \in \mathcal{F}$ with $x \in F \subset U$.

Theorem 12. *Let G be a rectifiable space, and let H be a rectifiable subspace of G compactly generated by a compact metrizable space F . Suppose further that $G = p(H, M)$, where M is a closed metrizable subspace of G . Then G is the union of a countable family of closed metrizable subspaces.*

Proof. By induction on n , we can define a sequence $\{A_n : n \in \omega\}$ of subsets of G such that:

- (1) $A_0 = F \cup p(F, F) \cup q(F, F)$;
- (2) $A_1 = p(A_0, A_0) \cup q(A_0, A_0)$;
- (3) $A_n = p(A_{n-1}, A_{n-1}) \cup q(A_{n-1}, A_{n-1})$.

Obviously, each $p(A_n, A_n), q(A_n, A_n), A_n$ are compact. Since compact space with a countable network is metrizable [10], it follows from Theorem 11 that each A_n is also metrizable. Since $H = \langle F \rangle$, $H = \bigcup_{n \in \omega} A_n$. Since $G = p(H, M)$, it follows from Theorem 11 again that G is the union of a countable family of closed metrizable subspaces. \square

A *neighborhood assignment* for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for each point $x \in X$. A space X is a *D-space*[6], if for any neighborhood assignment φ for X there is a closed discrete subset D of X such that $X = \bigcup_{d \in D} \varphi(d)$.

Corollary 9. *Let G be a rectifiable space, and let H be a rectifiable subspace of G compactly generated by a compact metrizable space F . Suppose further that $G = p(H, M)$, where M is a closed metrizable subspace of G . Then G is a D-space.*

Proof. It is well known that each metrizable space is a D -space. Hence M is a D -space, and then each $p(h, M)$ is a D -space, too. Since a countable infinite union of closed D -subspaces is D [3], it follows that $G = p(H, M) = \bigcup_{h \in H} p(h, M)$ is a D -space. \square

Recall that a space X has a *quasi- G_δ -diagonal* provided there is a sequence $\{\mathcal{G}(n) : n \in \mathbb{N}\}$ of collections of open subsets of X such that for any distinct points $x, y \in X$ there is a number n with $x \in st(x, \mathcal{G}(n)) \subset X \setminus \{y\}$.

Theorem 13. *Let G be a compactly generated Tychonoff rectifiable space, and $Y = bG \setminus G$ have locally quasi- G_δ -diagonal, where bG is a Hausdorff compactification of G . Then G satisfies one of the following conditions:*

- (1) G is locally compact;
- (2) G is separable and metrizable.

Proof. Suppose that G is nowhere locally compact. Since G is σ -compact, it follows from [14, Theorem 7.3] that G is separable and metrizable. \square

A space X is said to have a *regular G_δ -diagonal* if the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X \times X$.

Since a rectifiable space with a countable pseudocharacter has a regular G_δ -diagonal [14] and a paracompact space with a G_δ -diagonal is submetrizable [10], we have the following proposition.

Proposition 3. *If G is a compactly generated rectifiable space with a countable pseudocharacter, then G is submetrizable.*

Proposition 4. *Let G be a compactly generated Tychonoff rectifiable space with a countable pseudocharacter, and let $Y = bG \setminus G$ be Lindelöf, where bG is a Hausdorff compactification of G . Then G is separable and metrizable.*

Proof. Since $Y = bG \setminus G$ is Lindelöf, G is countable type [12], and thus G is a p -space [1]. Then G is a σ -compact p -space with a G_δ -diagonal, hence it is separable and metrizable [10, Corollaries 3.8 and 3.20]. \square

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